

SHIFT EQUIVALENCE AND A CATEGORY EQUIVALENCE INVOLVING GRADED MODULES OVER PATH ALGEBRAS OF QUIVERS

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ABSTRACT. In this paper we associate an abelian category to a finite directed graph and prove the categories arising from two graphs are equivalent if the incidence matrices of the graphs are shift equivalent. The abelian category is the quotient of the category of graded vector space representations of the quiver obtained by making the graded representations that are the sum of their finite dimensional submodules isomorphic to zero.

Actually, the main result in this paper is that the abelian categories are equivalent if the incidence matrices are strong shift equivalent. That result is combined with an earlier result of the author to prove that if the incidence matrices are shift equivalent, then the associated abelian categories are equivalent.

Given William's Theorem that subshifts of finite type associated to two directed graphs are conjugate if and only if the graphs are strong shift equivalent, our main result can be reformulated as follows: if the subshifts associated to two directed graphs are conjugate, then the categories associated to those graphs are equivalent.

1. INTRODUCTION

1.1. This paper proves a result of the following general type: if two graphs are equivalent in an appropriate sense, then certain algebraic objects associated to them are equivalent in a corresponding sense. We associate an abelian category to a directed graph and prove the categories arising from two graphs are equivalent if the graphs are equivalent in an appropriate sense.

A nice paper by Bates and Pask [3] contains results of this general type. They prove various isomorphisms and Morita equivalences between graph C^* -algebras that unify earlier results for graph C^* -algebras and Cuntz-Kreiger algebras (references for some of those earlier results can be found in the opening paragraph of [3]). Raeburn's monograph [7] is a comprehensive treatment of graph C^* -algebras. Analogous, but algebraic as opposed to

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C^* -algebraic, results for Leavitt path algebras have been proved by Abrams et al. See, for example, [1] and [2] and the references therein.

1.2. The graph equivalences alluded to in the opening paragraph of this introduction arise in the theory of subshifts of finite type.

A shift space (X, σ) over a finite alphabet \mathfrak{A} is a compact subset X of $\mathfrak{A}^{\mathbb{Z}}$ that is stable under the shift map σ defined by $\sigma(f)(n) = f(n+1)$.

To a directed graph Q with finitely many edges and no sources or sinks one may associate a shift space X_Q , called the *edge shift* of Q , whose alphabet is the set of arrows in Q (see [6, Defn. 2.2.5]). Edge shifts are subshifts of finite type.

One may also associate a subshift of finite type X_A to every square 0-1 matrix A having no zero rows or columns (see [6, Defn. 2.3.7]). The alphabet is now the set of vertices for the directed graph whose incidence matrix is A .

The equivalence between shift spaces that concerns us is *conjugacy*. Shift spaces (X, σ_X) and (Y, σ_Y) are *conjugate*, or *topologically conjugate*, denoted $X \cong Y$, if there is a homeomorphism $\phi : X \rightarrow Y$ such that $\sigma_Y \circ \phi = \phi \circ \sigma_X$.

By [6, Prop. 2.3.9], every subshift of finite type is conjugate to an edge shift X_Q for some directed graph Q .

One may associate to Q a new graph $Q^{[2]}$ (see [6, Defn. 2.3.4]) whose vertices are the edges of Q and $Q^{[2]}$ has an arrow from e to e' if e terminates where e' begins. The incidence matrix for $Q^{[2]}$, which we denote by B_Q , is a 0-1 matrix such that $X_Q \cong X_{B_Q}$. Thus every subshift of finite type is conjugate to a shift described by a 0-1 matrix. Conversely, every shift described by a 0-1 matrix A is conjugate to an edge shift: if Q^A is the directed graph with incidence matrix A , then X_{Q^A} is conjugate to X_Q [6, Exer. 1.5.6 and Prop. 2.3.9]. Hence subshifts of finite type are edge shifts or shifts associated to 0-1 matrices.

1.3. **The results.** Throughout k is a field and Q a directed graph, or quiver, with a finite number of vertices and arrows—loops and multiple arrows between vertices are allowed.

We write kQ for the path algebra of Q . The finite paths, including the trivial path at each vertex, in Q form a basis for kQ and multiplication is given by concatenation of paths.

We adopt the convention that the incidence matrix of Q is $C = (c_{ij})$ where

$$c_{ij} = \text{the number of arrows from } j \text{ to } i.$$

Given a square \mathbb{N} -valued matrix we write Q^C for the directed graph with incidence matrix C .

We make kQ an \mathbb{N} -graded algebra by declaring that a path is homogeneous of degree equal to its length. The category of \mathbb{Z} -graded left kQ -modules with degree-preserving homomorphisms is denoted by $\text{Gr}kQ$ and we write $\text{Fdim}kQ$ for its full subcategory of consisting of modules that are the sum of

their finite-dimensional submodules. Since $\text{Fdim} kQ$ is a Serre subcategory of $\text{Gr} kQ$ (it is, in fact, a localizing subcategory) we may form the quotient category

$$\text{QGr } kQ := \frac{\text{Gr } kQ}{\text{Fdim } kQ}.$$

The main results in this paper is the following theorem and its consequences.

Theorem 1.1. *Let L and R be \mathbb{N} -valued matrices such that LR and RL make sense. Let Q^{LR} be the quiver with incidence matrix LR and Q^{RL} the quiver with incidence matrix RL . There is an equivalence of categories*

$$\text{QGr } kQ^{LR} \equiv \text{QGr } kQ^{RL}.$$

Strong shift equivalence (see section 2.1 for its definition) is an equivalence relation on square matrices with entries in \mathbb{N} that is important in symbolic dynamics (see section 2.1 below). By interpreting a square matrix with entries in \mathbb{N} as an incidence matrix, an equivalence relation on square matrices with entries in \mathbb{N} is the same thing as an equivalence relation on finite directed graphs.

Theorem 1.2. *If the incidence matrices for Q and Q' are strong shift equivalent, then $\text{QGr}(kQ) \equiv \text{QGr}(kQ')$.*

Given William's Theorem (see Theorem 2.1 below), Theorem 1.2 can be restated as follows: Let (X, σ) and (X, σ') be subshifts of finite type, and Q and Q' directed graphs such that $(X, \sigma) = X_Q$ and $(X, \sigma') = X_{Q'}$. If (X, σ) and (X, σ') are conjugate, then the categories $\text{QGr}(kQ)$ and $\text{QGr}(kQ')$ are equivalent.

It is difficult to decide if two given matrices are strong shift equivalent. It is not known whether the strong shift equivalence problem is decidable. However, there is a weaker notion, shift equivalence (see section 2.4 for its definition), and Kim and Roush [4] have shown that the shift equivalence problem is decidable. Strong shift equivalence implies shift equivalence but the question of whether the two notions were the same was open for over twenty years before Kim and Roush [5] gave an example in 1999 showing shift equivalence does not imply strong shift equivalence.

If two incidence matrices A and B are shift equivalent, then A^ℓ is strong shift equivalent to B^ℓ for some integer ℓ . Theorem 1.8 in [9] says that if $Q^{(\ell)}$ is the directed graph whose incidence matrix is the ℓ^{th} power of the incidence matrix for Q , then $\text{QGr } kQ$ is equivalent to $\text{QGr}(kQ^{(\ell)})$.¹ Combining [9, Thm. 1.8] with Theorem 1.2 gives the following.

Corollary 1.3. *If the incidence matrices for Q and Q' are shift equivalent, then $\text{QGr}(kQ) \equiv \text{QGr}(kQ')$.*

¹In symbolic dynamics $Q^{(\ell)}$ is called the ℓ^{th} higher power graph of Q [6, Defn. 2.3.10].

Given Q , define $Q^{[n]}$ to be the following quiver: its vertices are the paths of length n in Q ; if p and q are paths of length n in Q there is an arrow in $Q^{[n]}$ from p to q if there is a path of length $n+1$ in Q that begins with p and ends with q .

Corollary 1.4. *For all integers $n \geq 2$, $\text{QGr}(kQ) \equiv \text{QGr}(kQ^{[n]})$.*

Acknowledgements. I wish to thank Doug Lind for introducing me to the notion of shift equivalence and for useful discussions about symbolic dynamics and related matters.

2. (STRONG) SHIFT EQUIVALENCE

2.1. Strong shift equivalence and Williams's Theorem. Let A and B be square matrices with entries in \mathbb{N} . An elementary strong shift equivalence between A and B is a pair of matrices L and R with non-negative integer entries such that

$$A = LR \quad \text{and} \quad B = RL.$$

We say A and B are **strong shift equivalent** if there is a chain of elementary strong shift equivalences from A to B .

The following fundamental result explains the importance of strong shift equivalence.

Theorem 2.1 (Williams). [10, Thm. A] *Let A and B be square \mathbb{N} -valued matrices and X_A and X_B the associated subshifts of finite type. Then $X_A \cong X_B$ if and only if A and B are strong shift equivalent.*

A proof of Theorem 2.1 can also be found at [6, Thm. 7.2.7].

2.2. The matrices

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix} \quad \text{and} \quad B = (2) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

are strong shift equivalent. The corresponding quivers are

$$Q^A = \begin{array}{ccc} \circ & \rightleftarrows & \circ \end{array} \quad Q^B = \begin{array}{ccc} \circ & \curvearrowright & \circ \end{array}$$

By Theorem 1.1, $\text{QGr } kQ^A \equiv \text{QGr } kQ^B$.

By [9], there are ultramatricial k -algebras $S(Q^A)$ and $S(Q^B)$ such that $\text{QGr } kQ^A \equiv \text{Mod } S(Q^A)$ and $\text{QGr } kQ^B \equiv \text{Mod } S(Q^B)$. The Bratteli diagram for $S(Q^A)$ is

$$\begin{array}{ccccccc} 1 & \longrightarrow & 2 & \longrightarrow & 4 & \longrightarrow & 8 & \longrightarrow & 16 & \longrightarrow & \cdots \\ & \searrow & \nearrow & & \searrow & \nearrow & & \searrow & \nearrow & & \\ 1 & \longrightarrow & 2 & \longrightarrow & 4 & \longrightarrow & 8 & \longrightarrow & 16 & \longrightarrow & \cdots \end{array}$$

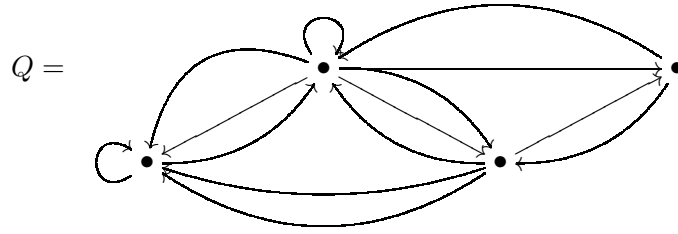
and that for $S(Q^B)$ is

$$1 \rightrightarrows 2 \rightrightarrows 4 \rightrightarrows 8 \rightrightarrows 16 \rightrightarrows \cdots$$

2.3. No general procedure is known to decide if two matrices are strong shift equivalent. The shortest known sequence of elementary strong shift equivalences proving that

$$\begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 6 \\ 1 & 1 \end{pmatrix}$$

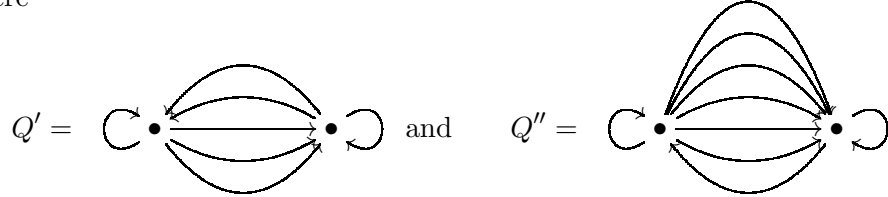
are strong shift equivalent was found by a computer search [6, Ex. 7.3.12] and en route from the first to the second matrix one passes through the incidence matrix for the graph



Thus Theorem 1.2 shows that

$$\text{QGr}(kQ) \equiv \text{QGr}(kQ') \equiv \text{QGr}(kQ'')$$

where



2.4. **Shift equivalence.** Two square matrices A and B with non-negative integer entries are **shift equivalent** if there is a positive integer ℓ and matrices L and R with non-negative integer entries such that

$$AL = LB, \quad RA = RB, \quad A^\ell = LR, \quad \text{and} \quad B^\ell = RL.$$

3. PROOF OF THEOREM 1.1

3.1. **Notation for quivers and path algebras.** Let k be a field and Q a finite quiver, i.e., a finite directed graph. We write Q_0 for its set of vertices and Q_1 for its set of arrows. If the arrow a ends where the arrow b starts we write ba for the path “first traverse a then traverse b ”. We write Q_n for the set of paths of length n in Q .

If p is a path we write $s(p)$ for the vertex at which it starts and $t(p)$ for the vertex at which it terminates.

The path algebra kQ has a basis given by the set of all finite paths, including the empty path and the trivial paths at each vertex. The multiplication

in kQ is the linear extension of that given by concatenation of paths, i.e.,

$$q \times p = \begin{cases} qp & \text{if } s(q) = t(p) \\ 0 & \text{if } s(q) \neq t(p). \end{cases}$$

The algebra kQ is \mathbb{N} -graded with degree n component equal to kQ_n , the linear span of the paths of length n . The subalgebra kQ_0 of kQ is isomorphic to a product of $|Q_0|$ copies of k and is therefore a semisimple ring. Each kQ_n is a kQ_0 -bimodule. The multiplication in kQ gives an isomorphism

$$(kQ_1)^{\otimes n} \cong kQ_n$$

of kQ_0 -bimodules where the tensor product on the left-hand side is taken over kQ_0 . It follows that kQ is isomorphic to the tensor algebra over kQ_0 of kQ_1 ,

$$kQ \cong T_{kI}(kQ_1).$$

3.2. Let i and j be positive integers.

Let kI denote the ring of k -valued functions on $[i] = \{1, \dots, i\}$ with pointwise addition and multiplication. Similarly, kJ denotes the ring of k -valued functions on $[j] = \{1, \dots, j\}$. We identify $kI \otimes_k kJ$ with the ring of k -valued functions on the Cartesian product $[i] \times [j]$. The category of kI - kJ -bimodules is equivalent to the category of $kI \otimes_k kJ$ -modules and we write E_{pq} for the simple kI - kJ -bimodule corresponding to $(p, q) \in [i] \times [j]$; i.e., E_{pq} is a copy of k supported at (p, q) .

3.3. Let $L = (\ell_{pq})$ be an $i \times j$ matrix over \mathbb{N} and $R = (r_{st})$ a $j \times i$ matrix over \mathbb{N} . Let Q^{LR} be the directed graph with incidence matrix LR and Q^{RL} the directed graph with incidence matrix RL .

Since it is unlikely to cause confusion we will also use the letter L to denote the kI - kJ -bimodule

$$L := \bigoplus_{\substack{p \in [i] \\ q \in [j]}} (E_{pq})^{\oplus \ell_{pq}}.$$

In a similar way we define the kJ - kI -bimodule

$$R := \bigoplus_{\substack{t \in [i] \\ s \in [j]}} (E_{st})^{\oplus r_{st}}.$$

The linear span in kQ^{LR} of the arrows in Q^{LR} is isomorphic to $L \otimes_{kJ} R$ as a kI - kI -bimodule. We identify the path algebras kQ and kQ' with the following tensor algebras:

$$kQ^{LR} = T_{kI}(L \otimes_{kJ} R) = \bigoplus_{n=0}^{\infty} (L \otimes_{kJ} R)^{\otimes n}$$

and

$$kQ^{RL} = T_{kJ}(R \otimes_{kI} L) = \bigoplus_{n=0}^{\infty} (R \otimes_{kI} L)^{\otimes n}.$$

We give kQ^{LR} its standard grading by declaring that kI is its degree-zero component and $L \otimes_{kJ} R$ its degree-one component.

3.4. Since kQ^{LR} is the tensor algebra of the kI -bimodule $L \otimes_{kJ} R$, a graded left kQ^{LR} -module is a pair (M, λ) consisting of a graded left kI -module M and a homomorphism

$$\lambda : L \otimes_{kJ} R \otimes_{kI} M \rightarrow M$$

of left kI -modules such that $\lambda(L \otimes_{kJ} R \otimes_{kI} M_n) \subset M_{n+1}$ for all n . A homomorphism $(M, \lambda) \rightarrow (M', \lambda')$ of graded kQ^{LR} -modules is a homomorphism $\theta : M \rightarrow M'$ of graded kI -modules such that

$$\theta \circ \lambda = \lambda' \circ (\text{id}_L \otimes \text{id}_R \otimes \theta).$$

3.5. We now define functors

$$\begin{array}{ccc} \text{Gr}(kQ^{LR}) & \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{F'} \end{array} & \text{Gr}(kQ^{RL}). \end{array}$$

If M is a graded left kQ^{LR} -module we define

$$F(M, \lambda) := (R \otimes_{kI} M, \text{id}_R \otimes \lambda)$$

with the grading

$$(R \otimes_{kI} M)_n := R \otimes_{kI} M_n.$$

The action of the degree-one component of kQ^{RL} , which is $R \otimes_{kI} L$, on the degree n component $F(M, \lambda)_n$ is

$$\begin{aligned} (\text{id}_R \otimes \lambda)((R \otimes_{kI} L) \otimes_{kJ} (R \otimes_{kI} M)_n) &= R \otimes_{kI} \lambda(L \otimes_{kJ} R \otimes_{kI} M_n) \\ &\subset R \otimes_{kI} M_{n+1} \\ &= (R \otimes_{kI} M)_{n+1} \end{aligned}$$

so $R \otimes_{kI} M$ really is a *graded* kQ' -module.

If $\theta : (M, \lambda) \rightarrow (M', \lambda')$ is a homomorphism of graded kQ^{LR} -modules we define

$$F(\theta) := \text{id}_R \otimes \theta.$$

It is easy to check that $F\theta : F(M, \lambda) \rightarrow F(M', \lambda')$ is a homomorphism of graded kQ^{RL} -modules. Hence F is a functor.

The functor F' is defined in a similar way.

Since kI and kJ are semisimple rings F and F' are exact functors.

Theorem 3.1. *Let L be an $i \times j$ matrix over \mathbb{N} and R a $j \times i$ matrix over \mathbb{N} . Then the functors F and F' induce mutually quasi-inverse equivalences of categories*

$$\text{QGr}(kQ^{LR}) \equiv \text{QGr}(kQ^{RL}).$$

Proof. Let $(M, \lambda) \in \text{Gr}(kQ^{LR})$. Then

$$\begin{aligned} F'F(M, \lambda) &= F'(R \otimes_{kI} M, \text{id}_R \otimes \lambda) \\ &= (L \otimes_{kJ} R \otimes_{kI} M, \text{id}_L \otimes \text{id}_R \otimes \lambda). \end{aligned}$$

We define $\tau_M : F'FM \rightarrow M$ by $\tau_M(x \otimes y \otimes m) := \lambda(x \otimes y \otimes m)$, Since $\tau_M = \lambda$ it is a tautology that

$$\tau_M \circ (\text{id}_L \otimes \text{id}_R \otimes \lambda) = \lambda \circ (\text{id}_L \otimes \text{id}_R \otimes \tau_M)$$

whence τ_M is a homomorphism of kQ^{LR} -modules. Since

$$(F'FM)_n = L \otimes_{kJ} (FM)_{n-1} = L \otimes_{kJ} R \otimes_{kI} M_{n-1}$$

we have

$$\tau_M((F'FM)_n) = \lambda(L \otimes_{kJ} R \otimes_{kI} M_{n-1}) \subset M_n.$$

Hence τ_M is a homomorphism of graded kQ^{LR} -modules.

The above shows that

$$\tau : F'F \rightarrow \text{id}_{\text{Gr}(kQ^{LR})}$$

is a natural transformation.

Since F and F' are exact functors that send finite dimensional modules to finite dimensional modules they induce functors between the quotient categories $\text{QGr}(kQ^{LR})$ and $\text{QGr}(kQ^{RL})$, say f and f' . It follows that τ induces a natural transformation from $f'f$ to $\text{id}_{\text{QGr}(kQ^{LR})}$. We will now show this induced natural transformation is an isomorphism of functors. A similar argument will show ff' is isomorphic to $\text{id}_{\text{QGr}(kQ^{RL})}$. The proof of the theorem will then be complete.

Write $V = L \otimes_{kJ} R$.

Claim: If $M \in \text{Gr}(kQ^{LR})$, then $F'FM \cong (kQ^{LR})_{\geq 1} \otimes_{kQ^{LR}} M$ as graded left kQ^{LR} -modules. Proof: Let (M, λ) be a graded left kQ^{LR} -module. Then

$$F'F(M, \lambda) = (V \otimes_{kI} M, \text{id}_V \otimes \lambda).$$

We make the identification $(kQ^{LR})_{\geq 1} = V \otimes_{kI} kQ^{LR}$ so the formula

$$\theta(v \otimes a \otimes m) := v \otimes am$$

for $v \in V$, $a \in kQ^{LR}$, and $m \in M$, defines an isomorphism of left kI -modules

$$\theta : (kQ^{LR})_{\geq 1} \otimes_{kQ^{LR}} M = V \otimes_{kI} kQ^{LR} \otimes_{kQ^{LR}} M \longrightarrow V \otimes_{kI} M.$$

If $v' \in V$ and $v \otimes a \otimes m \in V \otimes_{kI} kQ^{LR} \otimes_{kQ^{LR}} M$, then

$$\begin{aligned} \theta(v' \cdot (v \otimes a \otimes m)) &= \theta(v' \otimes va \otimes m) \\ &= v' \otimes vam \\ &= (\text{id}_V \otimes \lambda)(v' \otimes v \otimes am) \\ &= v' \cdot (v \otimes am) \\ &= v' \cdot \theta(v \otimes a \otimes m) \end{aligned}$$

so θ is a homomorphism, and therefore an isomorphism, of left kQ^{LR} -modules. In fact,

$$\theta : (kQ^{LR})_{\geq 1} \otimes_{kQ^{LR}} M \rightarrow F'F(M, \lambda)$$

is an isomorphism of *graded* kQ -modules because if $v \otimes a \otimes m$ is a homogeneous element of $V \otimes_{kI} kQ \otimes_{kQ} M = (kQ)_{\geq 1} \otimes_{kQ} M$, then

$$\deg(v \otimes a \otimes m) = 1 + \deg(am);$$

however, $F'F(M, \lambda)_n = V \otimes_{kI} M_{n-1}$ so, as an element of $F'F(M, \lambda)$, $\deg(v \otimes am) = 1 + \deg(am)$ so

$$\deg \theta(v \otimes a \otimes m) = \deg(v \otimes am) = \deg(v \otimes a \otimes m);$$

i.e., θ is a degree-preserving map so an isomorphism in $\text{Gr } kQ$. This completes the proof of the claim. \diamond

The claim shows that the homomorphisms

$$\begin{aligned} \eta_M : F'F(M, \lambda) &= (V \otimes_{kI} M, \text{id}_V \otimes \lambda) \rightarrow (kQ)_{\geq 1} \otimes_{kQ} M \\ \eta_M(v \otimes m) &= v \otimes m \end{aligned}$$

produce an isomorphism of functors

$$\eta : F'F \rightarrow (kQ)_{\geq 1} \otimes_{kQ} -.$$

Consider the diagram

$$\begin{array}{ccccccc} & & F'FM & \xrightarrow{\tau_M} & M & & \\ & & \downarrow \eta_M & & \parallel & & \\ 0 & \longrightarrow & \text{Tor}_1^{kQ}(kI, M) & \longrightarrow & (kQ)_{\geq 1} \otimes_{kQ} M & \xrightarrow{\mu} & M \longrightarrow kI \otimes_{kQ} M \longrightarrow 0 \end{array}$$

where the bottom row is the exact sequence obtained by applying $- \otimes_{kQ} M$ to the exact sequence of kQ -bimodules $0 \rightarrow (kQ)_{\geq 1} \rightarrow kQ \rightarrow kI \rightarrow 0$ and μ is the multiplication in kQ . Since $\tau_M(v \otimes m) = \lambda(v \otimes m) = vm$, the square commutes. Hence

$$\ker \tau_M \cong \text{Tor}_1^{kQ}(kI, M) \quad \text{and} \quad \text{coker } \tau_M \cong kI \otimes_{kQ} M.$$

Both these modules are annihilated by $(kQ)_{\geq 1}$ so belong to $\text{Fdim}(kQ)$. Therefore, after passing to $\text{QGr } kQ$, the diagram yields a commutative square in which μ and η_M become isomorphisms. It follows that τ_M becomes an isomorphism in $\text{QGr } kQ$, and hence that $\tau : F'F \rightarrow \text{id}_{\text{QGr } kQ}$ is an isomorphism of functors as claimed.

Given the symmetry of the situation we can reverse the roles of L and R and repeat the previous argument to produce an isomorphism of functors $FF' \rightarrow \text{id}_{\text{QGr } kQ'}$. This completes the proof that $\text{QGr } kQ$ is equivalent to $\text{QGr } kQ'$. \square

4. PROOF OF THEOREM 1.2

Suppose the \mathbb{N} -valued matrices A and B are strong shift equivalent. By definition, there is a sequence of matrices

$$A = A_1, A_2, \dots, A_n = B$$

and elementary strong shift equivalences between A_i and A_{i+1} for $1 \leq i \leq n-1$. If Q^{A_i} is the quiver with incidence matrix A_i , then repeated applications of Theorem 1.1 show that

$$\text{QGr } kQ^{A_1} \equiv \text{QGr } kQ^{A_2} \equiv \dots \equiv \text{QGr } kQ^{A_n}$$

thereby proving Theorem 1.2.

4.1. In-splitting and out-splitting. I am grateful to Min Wu for telling me that Theorem 1.1 applies to in-splittings and out-splittings.

Lind and Marcus define and discuss in-splittings and out-splittings of a directed graph in section 2.4 of [6]. As the name suggests, in-splitting involves replacing one vertex v by several, say n , vertices v_1, \dots, v_n and replacing each arrow a ending at v by n arrows a_1, \dots, a_n where a_i starts where a does and ends at v_i . Actually, in-splitting is a more general process than this, but the basic idea is along the lines just described. Out-splitting is an analogous process, now based on the arrows leaving a vertex.

The important point for us is that if Q' is obtained from Q by an in-splitting or an out-splitting there is an elementary strong shift equivalence between their incidence matrices (see [6, Thm. 2.4.12] and [6, Exer. 2.4.9]). Theorem 1.1 therefore yields the following result.

Corollary 4.1. *If Q' is obtained from Q by an in-splitting or out-splitting, then $\text{QGr } kQ \equiv \text{QGr } kQ'$.*

5. PROOF OF COROLLARY 1.4

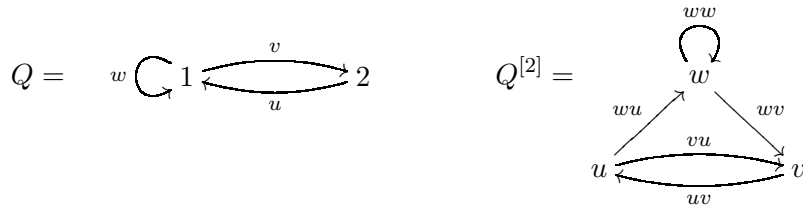
5.1. Let Q be a finite quiver. Define $Q^{[2]}$ by setting

$$Q_0^{[2]} := Q_1 \quad \text{and} \quad Q_1^{[2]} := Q_2$$

with a length-two path ba in Q being considered as an arrow in $Q^{[2]}$ from a to b .

If a and b are arrows in Q , there is at most one arrow in $Q^{[2]}$ from a to b so the incidence matrix for $Q^{[2]}$ is a 0-1 matrix.

The following example from [7, Example 2.7] and [6, Example 1.4.2] illustrates the construction:



Returning to the general case, define a $|Q_0| \times |Q_1|$ matrix L by

$$L_{ia} = \begin{cases} 1 & \text{if } t(a) = i \\ 0 & \text{if } t(a) \neq i \end{cases}$$

for $i \in Q_0$ and $a \in Q_1$, and a $|Q_1| \times |Q_0|$ matrix R by

$$R_{ai} = \begin{cases} 1 & \text{if } s(a) = i \\ 0 & \text{if } s(a) \neq i. \end{cases}$$

Then LR is the $|Q_0| \times |Q_0|$ matrix with entries

$$(LR)_{ij} = |\{a \in Q_1 \mid t(a) = i \text{ and } s(a) = j\}| \\ = \text{the number of arrows in } Q \text{ from } j \text{ to } i$$

and RL is the $|Q_1| \times |Q_1|$ matrix with entries

$$(RL)_{ab} = |\{i \in Q_0 \mid s(a) = i = t(b)\}| = \begin{cases} 1 & \text{if } s(a) = t(b) \\ 0 & \text{if } s(a) \neq t(b). \end{cases}$$

Therefore LR is the incidence matrix for Q and RL is the incidence matrix for $Q^{[2]}$. Theorem 1.1 therefore gives an equivalence

$$(5-1) \quad \text{QGr } kQ \equiv \text{QGr } kQ^{[2]}.$$

5.2. Let Q be a finite quiver and define $Q^{[n]}$ by setting

$$Q_0^{[n]} := Q_n \quad \text{and} \quad Q_1^{[n]} := Q_{n+1}$$

with a path $a_n \dots a_0$ of length $n+1$ in Q being considered as an arrow in $Q^{[n]}$ from $a_{n-1} \dots a_0$ to $a_n \dots a_1$.

It is an easy exercise to show that

$$\left(Q^{[n-1]}\right)^{[2]} = Q^{[n]}.$$

Repeatedly applying the equivalence in (5-1) gives a chain of equivalences

$$\text{QGr } kQ \equiv \text{QGr } kQ^{[2]} \equiv \dots \equiv \text{QGr } kQ^{[n]}$$

thereby proving Corollary 1.4.

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